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A Note Concerning Subspaces Invariant under an Incidence Matrix

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Several results have been obtained giving necessary conditions for the existence of a projective plane of order n [*1*], of a λ, k, v design [*2*], of a cyclic plane [*2*], and of certain collineations of a plane [*4*]. The proofs given in these papers are similar in that they depend on the results of Hasse-Minkowski on the rational equivalence of quadratic forms. However, aside from the use of quadratic-form theory, they differ in both method and detail. The purpose of this note is to show that the above results stem from the application of a single idea involving quadratic forms. Although the method is equally applicable to λ, k, v designs, we will, for simplicity of exposition, limit ourselves to projective planes.

Before stating the theorem on quadratic forms which we need, we recall some definitions and facts. Let V be a finite-dimensional vector space over a field F, and let q be a nondegenerate quadratic form on V to F. S is a similarity transformation on V relative to q, with norm $(S) = n$, if S is an F-automorphism of V and $q(Sv) = nq(v)$ for all $v \in V$. The discriminant of q is an element of $F^*/(F^*)^2$, the multiplicative group of F modulo the subgroup of squares. We use disc q to denote a pre-image in F^* of the discriminant of q. Let Q_p be the (complete) field of p-adic rationals. The Hilbert symbol, $(\ ,\ )_p$, on Q_p may be defined as follows:

For $\alpha, \beta \in Q_p$, $\alpha\beta \neq 0$,

$$(\alpha, \beta)_p = \begin{cases} 1 & \text{if there exist } x, y \in Q_p \text{ such that} \quad \alpha x^2 + \beta y^2 = 1 \\ -1 & \text{otherwise.} \end{cases}$$

The following is the theorem we use, a proof of which may be found in [*3*] or [*5*].

THEOREM. *Let V be a vector space of dimension m over the rational field Q, and let q be a nondegenerate quadratic form on V to Q. A necessary condition* *that there exists a similarity transformation on V relative to q with norm n is that, for all primes p,*

$$(n, (-1)^{m(m+1)/2} (\operatorname{disc} q)^{m+1})_p = 1.$$

We recall some facts concerning projective planes. An exposition of these results may be found in [7]. If π is a projective plane of order n, then π has $N = n^2 + n + 1$ points and N lines, every point of π lies on $n + 1$ lines, and every line of π contains $n + 1$ points. An incidence matrix for π, relative to orderings of the points and of the lines of π, is an N by N matrix $A = (a_{ij})$ where $a_{ij} = 1$ if the ith point lies on the jth line and $a_{ij} = 0$ otherwise. If A is an incidence matrix for a projective plane of order n and if ${}^tA$ denotes the transpose of A, then ${}^tAA = nI + J$ where I is the N by N identity matrix and J is the N by N matrix each of whose entries is 1.

Let E be a Euclidean N-dimensional space over Q, let e denote the (Euclidean) quadratic form on E, and let $(\,|\,)$ denote the bilinear form associated with e. $N \times N$ matrices will be considered as linear transformations on E. A simple consequence of the above theorem is the following.

PROPOSITION. *Let A be an incidence matrix for a projective plane of order n. A necessary condition that there exists an m-dimensional A-subspace, W, of E which is annihilated by J (i.e., $Jw = 0$ for all $w \in W$) is that, for all primes p,*

$$(n, (-1)^{m(m+1)/2} (\operatorname{disc} e\,|_W)^{m+1})_p = 1,$$

where $e\,|_W$ is the restriction of e to W.

Proof. Let $(\,|\,)_W$ denote the bilinear form associated with $e\,|_W$. For all $w_1, w_2 \in W$.

$$\begin{aligned}(Aw_1 \mid Aw_2)_W &= ({}^tAAw_1 \mid w_2)_W = ((nI + J)\, w_1 \mid w_2)_W \\ &= (nw_1 \mid w_2)_W = n(w_1 \mid w_2)_W .\end{aligned}$$

Hence A induces a similarity transformation of norm n on W. An application of the above theorem yields the proposition.

All the results noted in the introductory paragraph can be obtained by choosing appropriately an A-subspace, W, of E, and then determining $\operatorname{disc} e\,|_W$. We give two illustrations of this process.

J is a symmetric matrix of rank 1. Hence the kernel of J is an $(N - 1)$ dimensional subspace, W, of E. Since J and the incidence matrix, A, of a projective plane of order n commute, W is an A-subspace of J. From $E = W \perp W^{\perp}$, $W^{\perp}$ the orthogonal complement of W, we get that

$$1 = \operatorname{disc} e = \operatorname{disc} e\,|_W \cdot \operatorname{disc} e\,|_{W^{\perp}} .$$

But $W^{\perp}$ is a one dimensional subspace of E, which is spanned by the vector each of whose entries is 1 (i.e., $W^{\perp}$ is spanned by a characteristic vector of J belonging to the characteristic value N of J). Hence $\operatorname{disc} e\,|_{W^{\perp}} = N$, and we may choose $\operatorname{disc} e\,|_{W} = N$. If we apply the Proposition and note that $N = n^2 + n + 1$ is always odd, we find that a necessary condition for the existence of a projective plane of order n is that, for all primes p,

$$(n, (-1)^{n(n+1)/2} N)_p = 1.$$

For all $a, b, c \in Q_p$, $(a, bc)_p = (a, b)_p (a, c)_p$. Furthermore, for all p, $(n, n^2 + n + 1)_p = 1$. Hence we have:

THEOREM (Bruck-Ryser). *A necessary condition for the existence of a projective plane of order n is that, for all primes p,*

$$(n, (-1)^{n(n+1)/2})_p = 1.$$

We now show how the Proposition may be used to prove a slight generalization of theorems of Hughes [4], and Chowla-Ryser [2].

THEOREM. *Let π be a projective plane of order n. A necessary condition that π have a collineation which permutes the points of π in t cycles of lengths $l_1, l_2, \ldots, l_t$, $l_i \geqslant 1$, is that, for all primes p,*

$$(n, (-1)^{(1/2)[(N-t)(N-t+1)]} (l_1 l_2 \cdots l_t)^{N-t+1})_p = 1.$$

Proof. Assume that π has a collineation satisfying the conditions in the theorem. Parker [6] has shown that the permutation of the points under any collineation, φ, of a finite projective plane and the permutation of the lines under φ are conjugate as elements of the symmetric group. Hence we may assume, without loss, that the incidence matrix, A, for the plane π is such that ${}^tPAP = A$, where P is the direct sum of t cyclic permutation matrices of orders $l_1, l_2, \ldots, l_t$. P has the characteristic root 1 with multiplicity t. Let V be the maximal subspace of E on which P (and hence also ${}^tP = P^{-1}$) acts as the identity. Then $\dim V = t$. Since P commutes with both A and J, V is invariant under both A and J. To determine $\operatorname{disc} e\,|_V$ we note first that V has a basis consisting of characteristic vectors of P associated with the characteristic value 1. From the direct sum representation of P we find that V has an orthogonal basis consisting of t vectors of the form

$$0 \oplus \cdots \oplus 0 \oplus \underbrace{(1, 1, \ldots, 1)}_{l_i} \oplus 0 \oplus \cdots \oplus 0,$$

$i = 1, 2, \ldots, t$. Hence $\operatorname{disc} e\,|_V = l_1 l_2 \cdots l_t$.

Now let $W = V^{\perp}$, the orthogonal complement of V in E. Then $\dim W = N - t$. From $1 = \operatorname{disc} e = \operatorname{disc} e|_V \cdot \operatorname{disc} e|_W$ it follows that we may choose $\operatorname{disc} e|_W = l_1 l_2 \cdots l_t$. W is an A-subspace of E. In fact, if $v \in V$ then $A^{-1}v \in V$ and $Jv \in V$, whence for all $v \in V$, $w \in W$,

$$\begin{aligned}(v \mid Aw) &= (AA^{-1}v \mid Aw) = (Av' \mid Aw) = ({}^{t}AAv' \mid w)\\ &= ((nI + J)\, v' \mid w) = (v'' \mid w) = 0,\end{aligned}$$

and $Aw \in W$. W is also a J-subspace of E. For if $v \in V$, $w \in W$, then $(v \mid Jw) = (Jv \mid w) = (v' \mid w) = 0$ and $Jw \in W$. Since the one dimensional J-subspace of E associated with the unique nonzero characteristic value of J is contained in V, it follows that J annihilates W.

We have produced an A-subspace, W, of E that is annihilated by J, with $\dim W = N - t$ and $\operatorname{disc} e|_W = l_1 l_2 \cdots l_t$. If we apply the Proposition, we find that it is necessary that, for all primes p,

$$(n, (-1)^{(1/2)[(N-t)(N-t+1)]}\, (l_1 l_2 \cdots l_t)^{N-t+1})_p = 1.$$

This proves the theorem.

COROLLARY (Hughes). *Let π be a projective plane of order n. A necessary condition that π have a collineation of odd prime order r that fixes an even number of points of π, is that, for all primes p,*

$$(n, (-1)^{(r-1)/2}\, r)_p = 1.$$

Proof. Suppose that π has a collineation satisfying the conditions in the corollary. Let f equal the number of points of π fixed by this collineation. Then, in the notation of the theorem, $t = f + (N - f)/r$, $l_1 = l_2 = \cdots = l_f = 1$, and $l_{f+1} = \cdots = l_t = r$. If we apply the theorem and note that N and r are odd, and that f is even, we find that it is necessary that, for all primes p,

$$(n, (-1)^{(r-1)/2}\, r)_p = 1.$$

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